Strategies for Whole-Number Computation

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Chapter 4

STRATEGIES FOR WHOLE-NUMBER COMPUTATION

Much of the public sees computational skill as the hallmark of what it means to know mathematics at the elementary school level. Although this is far from the truth, the issue of computational skills with whole numbers is, in fact, a very important part of the elementary curriculum, especially in grades 2 to 6.

Rather than constant reliance on a single method of subtracting (or any operation), computational methods can and should change flexibly as the numbers and the context change. In the spirit of the *Standards,* the issue is no longer a matter of "knows how to subtract three-digit numbers"; rather it is the development over time of an assortment of flexible skills that will best serve students in the real world.

It is quite possible that you do not have these skills, but you can acquire them. Work at them as you learn about them. Equip yourself with a flexible array of computational strategies.

big ideas

1. **Flexible methods of computation involve taking apart and combining numbers in a wide variety of ways. Most of the partitions of numbers are based on place value or "compatible" numbers—number pairs that work easily together, such as 25 and 75.**
2. **Invented strategies are flexible methods of computing that vary with the numbers and the situation. Successful use of the strategies requires that they be understood by the one who is using them—hence, the term *invented.* Strategies may be invented by a peer or the class as a whole; they may even be suggested by the teacher. However, they must be constructed by the student.**
3. **Flexible methods for computation require a good understanding of the operations and properties of the operations, especially the turnaround property and the distributive property for multiplication. How the operations are related—addition to subtraction, addition to multiplication, and multiplication to division—is also an important ingredient.**
4. **The traditional algorithms are clever strategies for computing that have been developed over time. Each is based on performing the operation on one place value at a time with transitions to an adjacent position (trades, regrouping, "borrows," or "carries"). These algorithms work for all numbers but are often far from the most efficient or useful methods of computing.**

Toward Computational Fluency

With today's technology the need for doing tedious computations by hand has essentially disappeared. At the same time, we now know that there are numerous methods of computing that can be handled either mentally or with pencil-and-paper support. In most everyday instances, these alternative strategies for computing are easier and faster, can often be done mentally, and contribute to our overall number sense. The traditional algorithms (procedures for computing) do not have these benefits.

Consider the following problem.

Mary has 114 spaces in her photo album. So far she has 89 photos in the album. How many more photos can she put in before the album is full?

STOP **Try solving the photo album problem using some method other than the one you were taught in school. If you want to begin with the 9 and the 4, try a different approach. Can you do it mentally? Can you do it in more than one way? Work on this before reading further.**

Here are just four of many methods that have been used by students in the primary grades to solve the computation in the photo album problem:

- 89 + 11 is 100. 11 + 14 is 25.
- 90 + 10 is 100 and 14 more is 24 plus 1 (for 89, not 90) is 25.
- Take away 14 and then take away 11 more or 25 in all.
- 89, 99, 109 (that's 20). 110, 111, 112, 113, 114 (keeping track on fingers) is 25.

Strategies such as these can be done mentally, are generally faster than the traditional algorithms, and make sense to the person using them. Every day, students and adults resort to error-prone, traditional strategies when other, more meaningful methods would be faster and less susceptible to error. Flexibility with a variety of computational strategies is an important tool for successful daily living. It is time to broaden our perspective of what it means to compute.

Figure 4.1 lists three general types of computing. The initial, inefficient direct modeling methods can, with guidance, develop into an assortment of invented strategies that are flexible and useful. As noted in the diagram, many of these methods can be handled mentally, although no special methods are designed specifically for mental computation. The traditional pencil-and-paper algorithms remain in the mainstream curricula. However, the attention given to them should, at the very least, be debated.

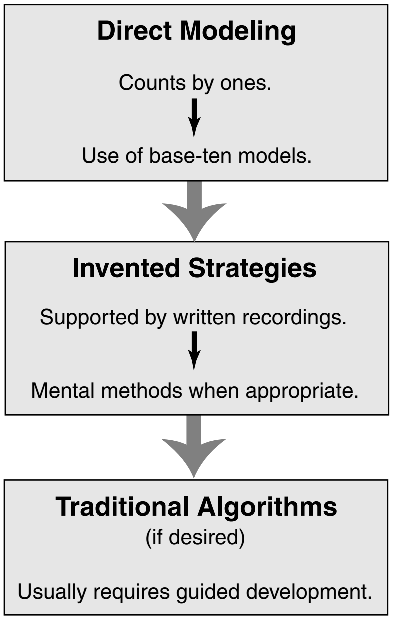


FIGURE 4.1

Three types of computational strategies.

Direct Modeling

The developmental step that usually precedes invented strategies is called *direct modeling:* the use of manipulatives or drawings along with counting to represent directly the meaning of an operation or story problem. Figure 4.2 provides an example using base-ten materials, but often students use simple counters and count by ones.

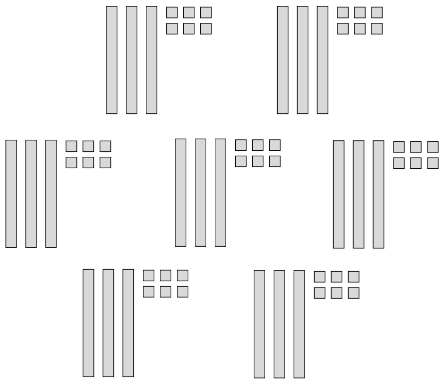


FIGURE 4.2

A possible direct modeling of 36 × 7 using base-ten models.

Students who consistently count by ones most likely have not developed base-ten grouping concepts. That does not mean that they should not continue to solve problems involving two-digit numbers. As you work with these children, suggest (don't force) that they group counters by tens as they count. Perhaps instead of making large piles, they might make bars of ten from connecting cubes or organize counters in cups of ten. Some students will use the ten-stick as a counting device to keep track of counts of ten, even though they are counting each segment of the stick by ones.

When children have plenty of experience with base-ten concepts and models, they begin to use these ideas in the direct modeling of the problems. Even when students use base-ten materials, they will find many different ways to solve problems.

Invented Strategies

We will refer to any strategy other than the traditional algorithm and that does not involve the use of physical materials or counting by ones as an *invented strategy*. These invented strategies might also be called *personal and flexible strategies*. At times, invented strategies are done mentally. For example, 75 + 19 can be done mentally (75 + 20 is 95, less 1 is 94). For 847 + 256, some students may write down intermediate steps to aid in memory as they work through the problem. (Try that one yourself.) In the classroom, some written support is often encouraged as strategies develop. Written records of thinking are more easily shared and help students focus on the ideas. The distinction between written, partially written, and mental is not important, especially in the development period.

Over the past two decades, a number of research projects have focused attention on how children handle computational situations when they have not been taught a specific algorithm or strategy. Three elementary curricula each base the development of computational methods on student-invented strategies. These are often referred to as "reform curricula" *(Investigations in Number, Data, and Space; Trailblazers;* and *Everyday Mathematics).* "There is mounting evidence that children both in and out of school can construct methods for adding and subtracting multi-digit numbers without explicit instruction" (Carpenter et al., 1998, p. 4). Data supporting students' construction of useful methods for multiplication and division have also been gathered (Baek, 1998; Fosnot & Dolk, 2001; Kamii & Dominick, 1997; Schifter, Bastable, & Russell, 1999b).

Not all students invent their own strategies. Strategies invented by class members are shared, explored, and tried out by others. However, no student should be permitted to use any strategy without understanding it.

Contrasts with Traditional Algorithms

There are significant differences between invented strategies and the traditional algorithms.

1. *Invented strategies are number oriented rather than digit oriented.* For example, one invented strategy for 68 × 7 begins 7 × 60 is 420 and 56 more is 476. The first product is 7 times *sixty,* not the digit 6, as would be the case in the traditional algorithm. Using the traditional algorithm for 45 + 32, children never think of 40

and 30 but rather 4 + 3. Kamii, long a crusader against standard algorithms, claims that they "unteach" place value (Kamii & Dominick, 1998).

2. *Invented strategies are left-handed rather than right-handed.* Invented strategies begin with the largest parts of numbers, those represented by the leftmost digits. For 26 × 47, many invented strategies begin with 20 × 40 is 800, providing some sense of the size of the eventual answer in just one step. The traditional algorithm begins with 7 × 6 is 42. By beginning on the right with a digit orientation, traditional methods hide the result until the end. Long division is an exception.
3. *Invented strategies are flexible rather than rigid.* Invented strategies tend to change with the numbers involved in order to make the computation easier. Try each of these mentally: 465 + 230 and 526 + 98. Did you use the same method? The traditional algorithm suggests using the same tool on all problems. The traditional algorithm for 7000 – 25 typically leads to student errors, yet a mental strategy is relatively simple.

Benefits of Invented Strategies

The development of invented strategies delivers more than computational facility. Both the development of these strategies and their regular use have positive benefits that are difficult to ignore.

- *Base-ten concepts are enhanced.* There is a definite interaction between the development of base-ten concepts and the process of inventing computational strategies (Carpenter et al., 1998). "Invented strategies demonstrate a hallmark characteristic of understanding" (p. 16). The development of invented strategies should be integrated with the development of base-ten concepts, even as early as first grade.
- *Invented strategies are built on student understanding.* Students rarely use an invented strategy they do not understand. In contrast, students are frequently seen to use traditional algorithms without being able to explain why they work (Carroll & Porter, 1997).
- *Students make fewer errors with invented strategies.* Data collected by Kamii and Dominick (1997) provide some hard evidence for this claim. With traditional algorithms, students tend to develop systematic errors or "buggy algorithms" that they use again and again. Careless errors often result from confusion with carried digits or column alignment. Systematic errors are not typical of invented strategies.
- *Invented strategies serve students at least as well on standard tests.* Evidence suggests that students not taught traditional algorithms fare about as well in computation on standardized tests as students in traditional programs (Campbell, 1996; Carroll, 1996, 1997; Chambers, 1996). As an added bonus, students tend to do quite well with word problems, since they are the principal vehicle for developing invented strategies. The pressures of external testing do not dictate a focus on the traditional algorithms.

Mental Computation

A mental computation strategy is simply any invented strategy that is done mentally. What may be a mental strategy for one student may require written support by another. Initially, students should not be asked to do computations mentally, as this may threaten those who have not yet developed a reasonable invented strategy or who are still at the direct modeling stage. At the same time, you may be quite amazed at the ability of students (and at your own ability) to do computations mentally.

Try your own hand with this example:

$$342 + 153 + 481$$

STOP **For the addition task just shown, try this method: Begin by adding the hundreds, saying the totals as you go—*3 hundred, 4 hundred, 8 hundred.* Then add on to this the tens in a successive manner and finally the ones. Do it now.**

When the computations are a bit more complicated, the challenge is more interesting and generally there are more alternatives. Here is an example taken from the grades 3–5 chapter of the *NCTM Standards* (p. 152).

$$7 \times 28$$

The *Standards* lists three paths to a solution but there are at least two more (NCTM, 2000, p. 152). How many ways can you find to do this one?

As your students become more adept, they can and should be challenged from time to time to do appropriate computations mentally. Do not expect the same skills of all students.

Traditional Algorithms

Teachers often ask, "How long should I wait until I show them the 'regular' way?" The question is based on a fear that without learning the same methods that all of us grew up with, students will somehow be disadvantaged. For addition and subtraction this is simply not the case. The primary goal for all computation should be students' ability to compute in some efficient manner—not what algorithms are used. That is, the *method* of computing is not the objective; the ability to compute is the goal. For multiplication and division, many teachers will see a greater need for traditional approaches, especially with three or more digits involved.

Abandon or Delay Traditional Algorithms

Flexible left-handed methods done mentally with written support are absolutely all that are necessary for addition and subtraction. Developed with adequate practice, these flexible approaches will become mental and very efficient for most students by fifth grade and will serve them more than adequately throughout life. You may find this difficult to accept for two reasons: first, because the traditional algorithms have been a significant part of your mathematical experiences and, second, because you may not have learned these skills. These are not reasons to teach the traditional algorithms for addition and subtraction.

For multiplication and division, the argument requires some discussion, especially as the number of digits involved increases. In the third grade, when students need only multiply or divide by a single digit, invented strategies are not only adequate but also will provide the benefits of understanding and flexibility mentioned earlier. The same types of skills used for two-digit numbers will be carried over to more complex computations as long as the focus remains on invented strategies and does not shift to the traditional algorithms. It is worth noting again that there is evidence that students do quite well on the computation portions of standardized tests even if they are never taught the traditional methods.

If, for whatever reason, you feel you must teach the traditional algorithms, consider the following:

- Students will not invent the traditional methods because right-handed methods are simply not natural. This means that you will have to introduce and explain each algorithm.
- No matter how carefully you suggest that these right-handed, borrow-and-carry, digit-oriented methods are simply another alternative, students will sense that these are the "right ways" or the "real ways" to compute. *This is how Mom and Dad do it. This is what the teacher taught us.* As a result, most students will abandon any flexible left-handed methods they may have been developing.

It is not that the traditional algorithms cannot be taught with a strong conceptual basis. Textbooks have been doing an excellent job of explaining these methods for years. The problem is that the traditional algorithms, especially for addition and subtraction, are not natural methods for students. As a result, the explanations generally fall on deaf ears. Far too many students learn them as meaningless procedures, develop error patterns, and require an excessive amount of reteaching or remediation. If you are going to teach the traditional algorithms, you are well advised to spend a significant amount of time—months, not weeks—with invented methods. Delay! The understanding that children gain from working with invented strategies will make it much easier for you to teach the traditional methods.

Traditional Algorithms Will Happen

You probably cannot keep the traditional algorithms out of your classroom. Children pick them up from older siblings, last year's teacher, or well-meaning parents. Traditional algorithms are in no way evil, and so to forbid their use is somewhat arbitrary. However, students who latch on to a traditional method often resist the invention of more flexible strategies. What do you do then?

First and foremost, apply the same rule to traditional algorithms as to all strategies: *If you use it, you must understand why it works and be able to explain it.* In an atmosphere that says, "Let's figure out why this works," students can profit from making sense of these algorithms just like any other. But the responsibility should be theirs, not yours.

Accept a traditional algorithm (once it is understood) as one more strategy to put in the class "tool box" of methods. But reinforce the idea that like the other strategies, it may be more useful in some instances than in others. Pose problems in which a mental strategy is much more useful, such as 504 – 498 or 75 × 4. Discuss which method seemed better. Point out that for a problem such as 4568 + 12,813, the traditional algorithm has some advantages. But in the real world, most people do those computations on a calculator.

Development of Invented Strategies: A General Approach

Students do not spontaneously invent wonderful computational methods while the teacher sits back and watches. Among different reform or progressive programs, students tended to develop or gravitate toward different strategies suggesting that teachers and the programs do have an effect on what methods students develop. This section discusses general pedagogical methods for helping children develop invented strategies.

Use Story Problems Frequently

When computational tasks are embedded in simple contexts, students seem to be more engaged than they are with bare computations. Furthermore, the choice of story problems influences the strategies students use to solve them. Consider these problems:

Max had already saved 68 cents when Mom gave him some money for running an errand. Now Max has 93 cents. How much did Max earn for his errand?

George took 93 cents to the store. He spent 68 cents. How much does he have left?

The computation 93 – 68 solves both problems, but the first is more likely than the second to be solved by an add-on method. In a similar manner, fair-share division problems are more likely to encourage a share strategy than a measurement or repeated subtraction problem.

Not every task need be a story problem. Especially when students are engaged in figuring out a new strategy, bare arithmetic problems are quite adequate.

Use the Three-Part Lesson Format

The three-part lesson format described in Chapter 1 is a good structure for an invented-strategy lesson. The task can be one or two story problems or even a bare computation but always with the expectation that the method of solution will be discussed.

Allow plenty of time to solve a problem. Listen to the different strategies students are using but do not interject your own. Challenge able students to find a second method, solve a problem without models, or improve on a written explanation. Allow children who are not ready for thinking with tens to use simple counting methods. Students who finish quickly may share their methods with others before sharing with the class.

The most important portion of the lesson comes when students explain their solution methods. Help students write their explanations on the board or overhead. Encourage students to ask questions of their classmates. Occasionally have the class try a particular method with different numbers to see how it works.

Remember, not every student will invent strategies. However, students can and will try strategies that they have seen and that make sense to them.

Select Numbers with Care

With traditional algorithms you are used to distinguishing between problems that require regrouping and those that do not. The number of digits involved is another common method of judging problem difficulty. When encouraging students to develop their own methods, there are more factors to consider. For addition, 35 + 42 is generally easier than 35 + 47. However, 30 + 20 is easier than both and can help students begin to think in terms of tens. A next step might be 46 + 10 or 20 + 63.

For subtraction, being able to give the other part of 100 is especially useful. Therefore, tasks such as *Thirty-five and how much more make 100?* can provide important readiness for later problems. Tasks such as 417 – 103 or 417 – 98 may each encourage students to subtract 100 and then adjust.

For multiplication, multiples of 5, 10, and 25 are good starting points. Even 325 × 4 may be easier than 86 × 7 even though there are three digits in the former example. For division, it is the divisor that requires attention. And, because most invented strategies for division rely on multiplication, the same comment applies. For example, 483 ÷ 75 is easier than 483 ÷ 67 and not much harder than 327 ÷ 6.

Integrate Computation with Place-Value Development

In Chapter 2 we made the point that as students develop computational strategies, they are enhancing their understanding of place value. Notice how the examples in the preceding section on number selection can help reinforce the way that our number system is built on a structure of groups of tens. In Chapter 2 there is a section entitled "Activities for Flexible Thinking with Whole Numbers" (pp. 51–56). The activities in that section are appropriate for grades 3 or 4 and complement the development of invented strategies, especially for addition and subtraction.

Progression from Direct Modeling

Direct modeling involving tens and ones can and will lead eventually to invented strategies. However, students may need to be encouraged to move away from the direct modeling process. Here are some ideas:

- Record students' verbal explanations on the board in ways that they and others can model. Have the class follow the recorded method using different numbers.
- Ask students who have just solved a problem with models to see if they can do it in their heads.
- Pose a problem to the class and ask students to solve it mentally if they are able.
- Ask children to make a written numeric record of what they did when they solved the problem with models. Explain that they are then going to try to use the same method on a new problem.

Invented Strategies for Addition and Subtraction

Research has demonstrated that children will invent a lot of different strategies for addition and subtraction. Your goal might be that each of your students has at least one or two methods that are reasonably efficient, mathematically correct, and useful with lots of different numbers. Expect different children to settle on different strategies.

It is not at all unreasonable for students to be able to add and subtract two-digit numbers mentally in the third grade. However, even in fourth grade, do not push all students to pure mental computation. By recording on the board the ideas that students use, you help all students develop new approaches. Those who need short-term memory assistance can see ways to support their strategies by jotting down intermediate results on paper. The goal should be flexible, meaningful computation. These methods tend to become mental with frequent use.

Most of the ideas suggested here for addition and subtraction can be taught and even mastered by the end of second grade. However, most third-grade students and even fifth- and sixth-grade students have not developed invented strategies. The sequence of ideas proposed is appropriate at any grade.

Adding and Subtracting Single Digits

Children can easily extend addition and subtraction facts to higher decades.

Tommy was on page 47 of his book. Then he read 8 more pages. How many pages did Tommy read in all?

If students are simply counting on by ones, the following activity may be useful. It is an extension of the make-ten strategy for addition facts.

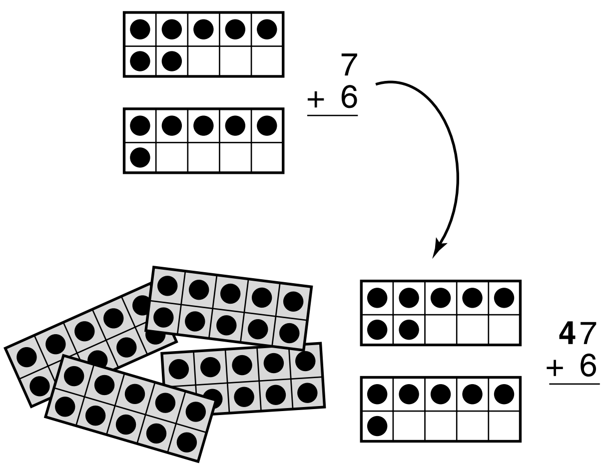


FIGURE 4.3

Little ten-frame cards can help students extend the make-ten idea to larger numbers.

ACTIVITY 4.1

Ten-Frame Adding and Subtracting

Quickly review the make-ten idea from addition facts using two ten-frames. (Add on to get up to ten and then add the rest.) Challenge students to use the same idea to add on to a two-digit number as shown in Figure 4.3. Two students can work together. First, they make a specified two-digit number with the little ten-frame cards. They then stack up all of the less-than-ten cards and turn them over one at a time. Together they talk about how to get the total quickly.

The same approach is used for subtraction. For instance, for 53 – 7, take off 3 to get to 50, then 4 more is 46.

Notice how building up through ten (as in 47 + 6) or down through ten (as in 53 – 7) is different from carrying and borrowing. No ones are exchanged for a ten nor a ten for ones. The ten-frame cards encourage students to work with multiples of ten without regrouping.

Another important model to use is the hundreds chart. The hundreds chart has the same tens structure as the little ten-frame cards. For 47 + 6 you count 3 to get out to 50 at the end of the row and then 3 more in the next row.

Adding Two-Digit Numbers

For each of the examples that follow, a possible recording method is offered. These are intended to be suggestions, not prescriptions. Students have difficulty inventing recording techniques. If you record their ideas on the board as they explain their ideas, you are helping them develop written techniques. You may even discuss recording methods with individuals or with the class to decide on a form that seems to work well. Horizontal formats encourage students to think in terms of numbers instead of digits. A horizontal format is also less likely to encourage use of the traditional algorithms.

Students will often use a counting-by-tens-and-ones technique for some of these methods. That is, instead of "46 + 30 is 76," they may count "46 ⟶ 56, 66, 76." These counts can be written down as they are said to help students keep track.

Figure 4.4 illustrates four different strategies for addition of two two-digit numbers. The following story problem is a suggestion.

The two Scout troops went on a field trip. There were 46 Girl Scouts and 38 Boy Scouts. How many Scouts went on the trip?

The *move to make-ten* and *compensation* strategies are useful when one of the numbers ends in 8 or 9. To promote that strategy, present problems with addends like 39 or 58. Note that it is only necessary to adjust one of the two numbers.

STOP **Try adding 367 + 155 in as many different ways as you can. How many of your ways are like those in Figure 4.4?**

Invented Strategies for Addition with Two-Digit Numbers	
Add Tens, Add Ones, Then Combine 46 + 38 40 and 30 is 70. 6 and 8 is 14. 70 and 14 is 84. 46 +38 = 70, 14, 84	**Move Some to Make Tens** 46 + 38 Take 2 from the 46 and put it with the 38 to make 40. Now you have 44 and 40 more is 84. 46 + 38 (2) 44 + 40 84
Add On Tens, Then Add Ones 46 + 38 46 and 30 more is 76. Then I added on the other 8. 76 and 4 is 80 and 4 is 84. 46 + 38 → 76 + 8 → 80, 84	**Use a Nice Number and Compensate** 46 + 38 46 and 40 is 86. That's 2 extra, so it's 84. 46 + 38 46 + 40 → 86 − 2 → 84

FIGURE 4.4

Four different invented strategies for adding two two-digit numbers.

Subtracting by Counting Up

This is an amazingly powerful way to subtract. Students working on the *think-addition* strategy for their basic facts can also be solving problems with larger numbers. The concept is the same. It is important to use *join with change unknown* problems or *missing-part* problems to encourage the counting-up strategy. Here is an example of each.

Sam had 46 baseball cards. He went to a card show and got some more cards for his collection. Now he has 73 cards. How many cards did Sam buy at the card show?

Juanita counted all of her crayons. Some were broken and some not. She had 73 crayons in all. 46 crayons were not broken. How many were broken?

The numbers in these problems are used in the strategies illustrated in Figure 4.5.

Emphasize the value of using tens by posing problems involving multiples of 10. In 50 – 17, the use of ten can happen by adding up from 17 to 20, or by adding 30 to 17. Some students may reason that it must be 30-something because 30 and 17 is less than 50, and 40 and 17 is more than 50. Because it takes 3 to go with 7 to make 10, the answer must be 33. Work on naming the missing part of 50 or 100 is also valuable. (See Activity 2.18, "The Other Part of 100," p. 54.)

Take-Away Subtraction

Take-away methods are more difficult to do mentally or even with the help of paper and pencil. This is especially true when problems involve three digits. Exceptions involve problems such as 423 – 8 or 576 – 300 (subtracting a number less than 10 or a multiple of 10 or 100). However, take-away strategies are bound to occur, probably because traditional textbooks emphasize take-away as the meaning of subtraction. Take-

FIGURE 4.5

Subtraction by counting up is a powerful method.

Invented Strategies for Subtraction by Counting Up

Add Tens to Get Close, Then Ones

73 – 46

46 and 20 is 66.
(30 more is too much.)
Then 4 more is 70 and 3 is 73.
That's 20 and 7 or 27.

46 > 20
66 > 4
70 > 3
73
27

Add Tens to Overshoot, Then Come Back

73 – 46

46 and 30 is 76.
That's 3 too much, so it's 27.

73 – 46
46 + 30 → 76 – 3 → 73
30 – 3 = 27

Add Ones to Make a Ten, Then Tens and Ones

73 – 46

46 and 4 is 50.
50 and 20 is 70 and 3 more is 73. The 4 and 3 is 7 and 20 is 27.

73 – 46
46 + 4 → 50
+ 20 → 70
+ 3 → 73
27

Similarly,

46 and 4 is 50.
50 and 23 is 73.
23 and 4 is 27.

46 + 4 → 50
50 + 23 → 73
23 + 4 = 27

away is very likely the strategy that will come to mind first for students who have previously been taught the traditional algorithm.

Four take-away strategies are shown in Figure 4.6, and these should not be discouraged. We suggest, however, that you emphasize adding-on methods whenever possible.

There were 73 children on the playground. The 46 third-grade students came in first. How many children were still outside?

The two methods that begin by taking tens from tens are reflective of what most students do with base-ten pieces. The other two methods leave one of the numbers intact and subtract from it. Try 83 – 29 in your head by first taking away 30 and adding 1 back. This is a good mental method when subtracting a number that is close to a multiple of ten.

STOP **Try computing 82 – 57. Use both take-away and counting up methods. Can you use all of the strategies in Figures 4.5 and 4.6 without looking?**

Extensions and Challenges

Each of the examples in the preceding sections involved sums less than 100 and all involved *bridging a ten;* that is, if done with a traditional algorithm, they require carrying or borrowing. Bridging, the size of the numbers, and the potential for doing problems mentally are all issues to consider.

Invented Strategies for Take-Away Subtraction

Take Tens from the Tens, Then Subtract Ones	Take Away Tens, Then Ones
73 – 46 70 minus 40 is 30. Take away 6 more is 24. Now add in the 3 ones ⟶ 27. 73 – 46 70 – 40 → 30 – 6 → 24 + 3 → 27	73 – 46 73 minus 40 is 33. Then take away 6: 3 makes 30 and 3 more is 27. 73 – 40 → 33 – 3 30 – 3 → 27
	Take Extra Tens, Then Add Back
Or	73 – 46 73 take away 50 is 23. That's 4 too many. 23 and 4 is 27. 73 – 50 → 23 + 4 27
70 minus 40 is 30. I can take those 3 away, but I need 3 more from the 30 to make 27. 73 – 46 30 – 3 27	**Add to the Whole If Necessary** 73 – 46 Give 3 to 73 to make 76. 76 take away 46 is 30. Now give 3 back ⟶ 27. +3 73 – 46 76 – 46 → 30 – 3 → 27

FIGURE 4.6

Take-away strategies work reasonably well for two-digit problems. They are a bit more difficult with three digits.

Bridging

For most of the strategies, it is easier to add or subtract when bridging is not required. Try each strategy with 34 + 52 or 68 – 24 to see how it works. Easier problems instill confidence. They also permit you to challenge your students with a "harder one." There is also the issue of bridging 100 or 1000. Try 58 + 67 with different strategies. Bridging across 100 is also an issue for subtraction. Problems such as 128 – 50 or 128 – 45 are more difficult than ones that do not bridge 100.

Larger Numbers

Most curricula will expect third graders to add and subtract three-digit numbers. Your state standards may even require work with four-digit numbers. Try seeing how *you* would do these without using the traditional algorithms: 487 + 235 and 623 – 247. For subtraction, a counting-up strategy is usually the easiest. Occasionally, other strategies appear with larger numbers. For example, "chunking off" multiples of 50 or 25 is often a useful method. For 462 + 257, pull out 450 and 250 to make 700. That leaves 12 and 7 more ⟶ 719.

Traditional Algorithms for Addition and Subtraction

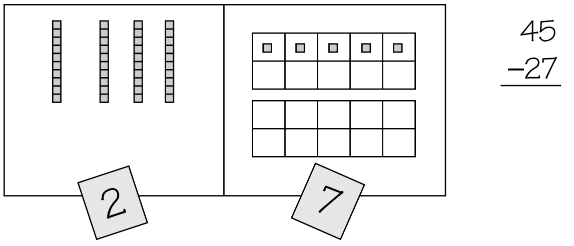
The traditional computational methods for addition and subtraction are significantly different from nearly every invented method. In addition to starting with the rightmost digits and being digit oriented (as already noted), the traditional approaches involve the concept generally referred to as *regrouping,* exchanging 10 in one place-value position for 1 in the position to the left ("carrying"), or the reverse, exchanging 1 for 10 in the position to the right ("borrowing"). The terms *borrowing* and *carrying* are obsolete and conceptually misleading. The word *regroup* also offers no conceptual help to students. A preferable term is *trade*. Ten ones are *traded* for a ten. A hundred is *traded* for 10 tens. Trading makes sense with the use of base-ten pieces when, in fact, pieces must be traded; for example, a ten piece is traded in for 10 ones pieces.

Terminology aside, the trading process is quite different from the bridging process used in all invented and mental strategies. Consider the task of adding 28 + 65. Using the traditional method, we first add 8 and 5. The resulting 13 ones are separated into 3 ones and 1 ten. The newly formed ten is then combined with the other tens. This process of "carrying a ten" is conceptually difficult and is different from the bridging process that occurs in invented strategies. In fact, nearly all major textbooks now teach this process of regrouping prior to and separate from direct instruction with the addition and subtraction algorithm, an indication of the difficulties involved. The process is even more difficult for subtraction, especially across a zero in the tens place where two successive trades are required.

Compounding all of this is the issue of recording each step. The traditional algorithms do not lend themselves to mental computation, so students must learn to record. The literature of the past 50 years is replete with the errors that students make with these recording methods.

All of these observations are offered to encourage you to abandon the traditional algorithms for addition and subtraction and, failing that, to alert you to the difficulties your students will likely have. Having said that, we offer some guidance for you if you must teach the standard procedures.

- Because it will never occur to students to add or subtract beginning in the ones place, you will have to use a more direct approach to instruction rather than a strictly problem-oriented approach.
- Use base-ten models and no recording at all until students seem to understand the process.
- For subtraction, model only the whole or top number. For the bottom number, have students write the digits on small slips of paper as shown in Figure 4.7.
- Develop the written method in a do-then-record approach. Whenever any change is made with the base-ten models, students record the action in the standard manner. Most traditional textbook explanations generally offer good guidance. However, they move to pure drill far too quickly with the result often being rules without reasons.
- Pay special attention to difficulties involving zero, especially in problems such as 504 – 347 where students must "borrow across zero." These problems should be solved with models and discussed as a class.



Not enough ones to take off 7.
Trade a ten for 10 ones.

FIGURE 4.7

Setting up the subtraction algorithm.

Invented Strategies for Multiplication

Computation strategies for multiplication are considerably more complex than for addition and subtraction. Often, but by no means always, the strategies that students invent are very similar to the traditional algorithm. The big difference is that students think about numbers, not digits. They always begin with the large or left-hand numbers.

For multiplication, the ability to break numbers apart in flexible ways is even more important than in addition or subtraction. The distributive property is another concept that is important in multiplication computation. For example, to multiply 43 × 5, one might think about breaking 43 into 40 and 3, multiplying each by 5, and then adding the results. Children require ample opportunities to develop these concepts by making sense of their own ideas and those of their classmates.

Useful Representations

The problem 34 × 6 may be represented in a number of ways, as illustrated in Figure 4.8. Often the choice of a model is influenced by a story problem. To determine how many Easter eggs 34 children need if each colors 6 eggs, children may model 6 sets of 34 (or possibly 34 sets of 6). If the problem is about the area of a rectangle that is 34 cm by 6 cm, then some form of an array is likely. But each representation is appropriate for thinking about 34 × 6 regardless of the context, and students should get to a point where they select ways to think about multiplication that are meaningful to them.

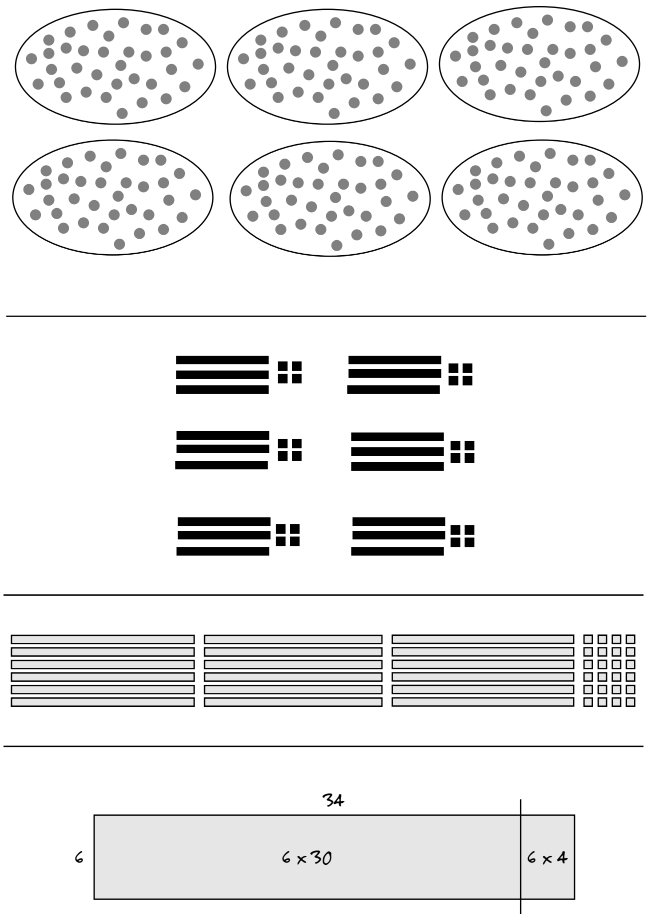


FIGURE 4.8

Different ways to model 34 x 6 may support different computational strategies.

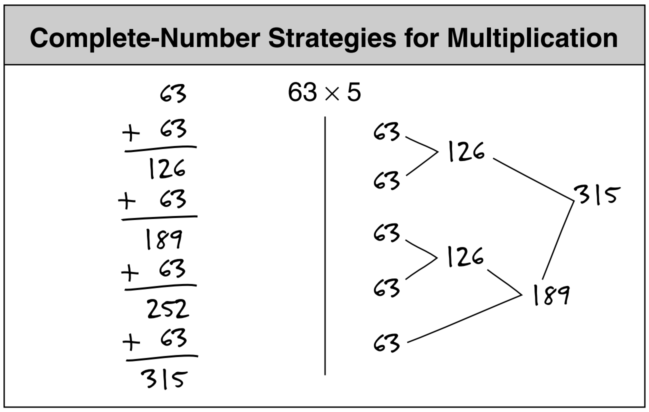


FIGURE 4.9

Children who use a complete-number strategy do not break numbers apart into decades or tens and ones.

How students represent a product interacts with their methods for determining answers. The groups of 34 might suggest repeated additions—perhaps taking the sets two at a time. Double 34 is 68 and there are three of those, so 68 + 68 + 68. From there a variety of methods are possible.

The six sets of base-ten pieces might suggest breaking the numbers into tens and ones: 6 times 3 tens or 6 × 30 and 6 × 4. Some children use the tens individually: 6 tens make 60. So that's 60 and 60 and 60 (180). Then add on the 24 to make 204.

It is not uncommon to arrange the base-ten pieces in a nice array, even if the story problem does not suggest it. The area model is very much like an arrangement of the base-ten pieces.

All of these ideas should be part of students' repertoire of models for multidigit multiplication. Introduce different representations (one at a time) as ways to explore multiplication until you are comfortable that the class has a collection of useful ideas. At the same time, do not force students who reason very well without drawings to use models when they are not needed.

Multiplication by a Single-Digit Multiplier

As with addition and subtraction, it is helpful to place multiplication tasks in contextual story problems. Let students model the problems in ways that make sense to them. Do not be concerned about mixing of factors (6 sets of 34 or 34 sets of 6). Nor should you be timid about the numbers you use. The problem 3 × 24 may be easier than 7 × 65, but the latter provides challenge. The types of strategies that students use for multiplication are much more varied than for addition and subtraction. However, the following three categories can be identified from the research to date.

Complete-Number Strategies

Students who are not yet comfortable breaking numbers into parts using tens and ones will approach the numbers in the sets as single groups. For students who think this way, Figure 4.9 illustrates two methods they may use. These children will benefit from listening to students who use base-ten models. They may also need more work with base-ten grouping activities where they take numbers apart in different ways.

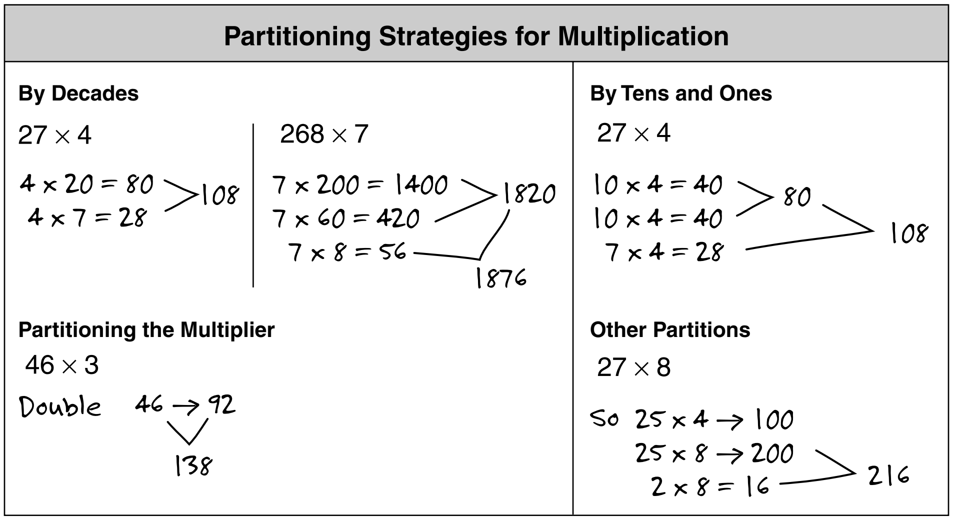


FIGURE 4.10

Numbers can be broken apart in different ways to make easier partial products, which are then combined. Partitioning by decades is useful for mental computation and is very close to the standard algorithm.

Partitioning Strategies

Students break numbers up in a variety of ways that reflect an understanding of base-ten concepts, at least four of which are illustrated in Figure 4.10. The "By Decades" approach is the same as the standard algorithm except that students always begin with the large values. It extends easily to three digits and is very powerful as a mental math strategy. Another valuable strategy for mental methods is found in the "Other Partitions" example. It is easy to compute mentally with multiples of 25 and 50 and then add or subtract a small adjustment. All partition strategies rely on the distributive property.

Compensation Strategies

Students look for ways to manipulate numbers so that the calculations are easy. In Figure 4.11, the problem 27 × 4 is changed to an easier one, and then an adjustment or compensation is made. In the second example, one factor is cut in half and the other doubled. This is often used when a 5 or a 50 is involved. Because these strategies are so dependent on the numbers involved, they can't be used for all computations. However, they are powerful strategies, especially for mental math and estimation.

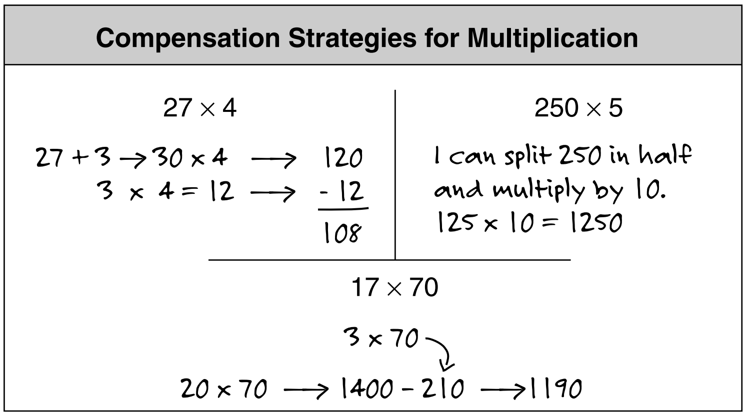


FIGURE 4.11

Compensation methods use a product related to the original. A compensation is made in the answer, or one factor is changed to compensate for a change in the other factor.

Using Multiples of 10 and 100

There is a value in exposing students early to products involving multiples of 10 and 100.

The Scout troop wanted to package up 400 fire starter kits as a fund-raising project. If each pack will have 12 fire starters, how many fire starters are the Scouts going to need?

Students will use 4 × 12 = 48 to figure out that 400 × 12 is 4800. There will be discussion around how to say and write "forty-eight hundred." Be aware of students who simply tack on zeros without understanding why. Try problems such as 30 × 60 or 210 × 40 where tens are multiplied by tens.

Two-Digit Multipliers

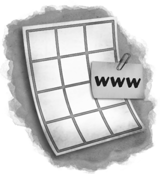
A problem such as this one can be solved in many different ways:

The parade had 23 clowns. Each clown carried 18 balloons. How many balloons were there altogether?

Some children look for smaller products such as 6 × 23 and then add that result three times. Another method is to do 20 × 23 and then subtract 2 × 23. Others will calculate four separate partial products: 10 × 20 = 200, 8 × 20 = 160, 10 × 3 = 30, and 8 × 3 = 24. And still others may add up a string of 23s. Two-digit multiplication is both complex and challenging. But students can solve these problems in a variety of interesting ways, many of which will contribute to the development of the traditional algorithm or one that is just as efficient. Time devoted to working on these tasks in the fourth and fifth grades is well spent.

Area Models

When working on multiplication strategies, a key idea is finding ways to break one or both of the numbers into smaller numbers. For 34 × 6, if 34 is broken into 30 and 4, both the 30 and the 4 must be multiplied by 6. Models are an enormous help in developing this idea. Refer again to Figure 4.8 (p. 114) to see models of 34 × 6. An area model can expand this idea to two-digit multipliers.



BLM 16

A valuable exploration is to prepare large rectangles for each group of two or three students. The rectangles should be measured carefully, with dimensions between 25 cm and 60 cm, and drawn accurately with square corners. (Use the corner of a piece of poster board for a guide.) The students' task is to determine how many small ones pieces (base-ten materials) will fit inside. Wooden or plastic base-ten pieces are best, but cardboard strips and squares are adequate. Alternatively, rectangles can be drawn on base-ten grid paper (see Blackline Masters).

Most students will fill the rectangle first with as many hundreds pieces as possible. One obvious approach is to put the 12 hundreds in one corner. This will leave narrow regions on two sides that can be filled with tens pieces and a final small rectangle that will hold ones. Especially if students have had earlier experiences with finding products in arrays, figuring out the size of each subrectangle is not terribly difficult. The sketch in Figure 4.12 shows the four regions.

If you did not already know the algorithm, how would you determine the size of the rectangle? Use your method (not the standard algorithm) on a rectangle that measures 68 cm × 24 cm. Make a sketch to show and explain your work.

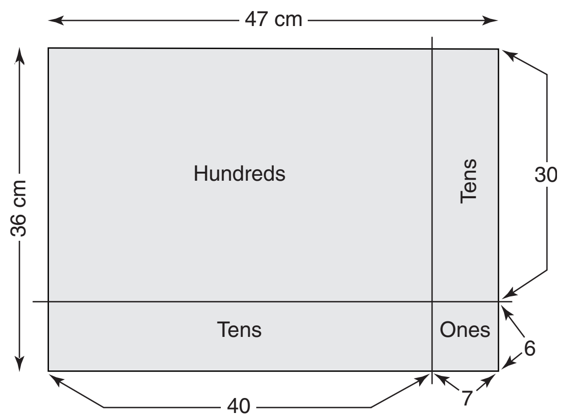


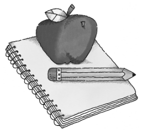
FIGURE 4.12

Ones, tens, and hundreds pieces fit exactly into the four sections of this 47 x 36 rectangle. Figure the size of each section to determine the size of the whole rectangle.

As you will see in the discussion of the traditional algorithm, the area model leads to a fairly reasonable approach to multiplying numbers, even if you never have students "carry," which is a source of many errors.

Cluster Problems

In the fourth and fifth grades of *Investigations in Number, Data, and Space* (one of the NSF-supported reform curricula), one approach to multidigit multiplication is called "cluster problems." This rather unique approach to the topic encourages students to use facts and combinations that they know or can easily figure out in order to find the answers to more complex computations. For example, the following cluster is used in an introductory lesson in the fourth-grade unit: 3 × 7, 5 × 7, 10 × 7, 50 × 7, and 53 × 7. The goal is to figure out the last product. Students solve all of the problems and explain what problems were helpful in solving the last problem. Not every problem in the cluster needs to be used to solve the final problem. If students wish to add other problems to the cluster to aid in finding their solution to the final problem, they are encouraged to do so.



EXPANDED LESSON

(pages 129–130)

The expanded lesson for this chapter has students work with the area model for two-digit multiplication.

Here are two cluster problems taken from a fourth-grade worksheet.

2 × 50	60 × 20
10 × 50	62 × 10
34 × 25	62 × 3
30 × 50	**62 × 23**
34 × 50	

It is useful to have students make an estimate of the final product before doing any of the problems in the cluster. In the first example cluster, 2 × 50 may be helpful in thinking about 10 × 50, which in turn is useful in knowing 30 × 50. Also 2 × 50 can be used to get 4 × 50. The results of 30 × 50 and 4 × 50 combine to give you 34 × 50. It may seem that 34 × 25 is harder than 34 × 50. However, if you know 34 × 25, it need only be doubled to get the desired product. Students should be encouraged to add problems to the cluster if they need them. Here is a good example: Think how you could use 10 × 34 (and some other related problems) to find 34 × 25.

The cluster-problem approach begins with students being provided with the cluster of problems. After they have become familiar with the approach, students should make up their own cluster of problems for a given product. At first, have students brainstorm clusters together as a class.

First, solve the two preceding clusters each in at least two ways. Now, try your hand at making up a cluster of problems for 86 × 42. Include all possible problems that you think might possibly be helpful, even if they are not all related to one approach to finding the product. Then use your cluster to find the product. Is there more than one way?

Here are some problems that might be in your cluster.

2 × 80 4 × 80 2 × 86 40 × 80 6 × 40 10 × 86 40 × 86

Of course, your cluster may have included products not shown here. All that is required to begin the cluster-problem approach is that your cluster eventually leads to a solution. Besides your own cluster, see if you can use the problems in this cluster to find 86 × 42.

Cluster problems help students think about ways that they can break numbers apart into easier parts. The strategy of breaking the numbers apart and multiplying the parts—the distributive property—is an extremely valuable technique for flexible computation. It is also fun to find different clever paths to the solution. For many problems, finding a workable cluster is actually faster than using an algorithm.

The Traditional Algorithm for Multiplication

The traditional multiplication algorithm is probably the most difficult of the four algorithms if students have not had plenty of opportunities to explore their own strategies. Time spent allowing your students to develop a range of invented strategies will pay off in their understanding of the traditional algorithm. While your students are working on multiplication using their invented strategies, be sure to emphasize partitioning techniques, especially those that are similar to the "By Decades" approach shown in Figure 4.10 (p. 115). These strategies tend to be the most efficient and are very close to the traditional algorithm. In fact, students who are using one or more partitioning strategies with a one-digit multiplier have no real need to learn any other approach.

The multiplication algorithm can be meaningfully developed using either a repeated addition model or an area model. For single-digit multipliers, the difference is minimal. When you move to two-digit multipliers, the area model has some advantages. For that reason, the discussion here will use the area model.

One-Digit Multipliers

As with the other algorithms, as much time as possible should be devoted to the conceptual development of the algorithm with the recording or written part coming later.

Begin with Models

Give students a drawing of a rectangle 47 cm by 6 cm. *How many small square centimeter pieces will fit in the rectangle?* (What is the area of the rectangle in square centimeters?) Let students solve the problem in groups before discussing it as a class.

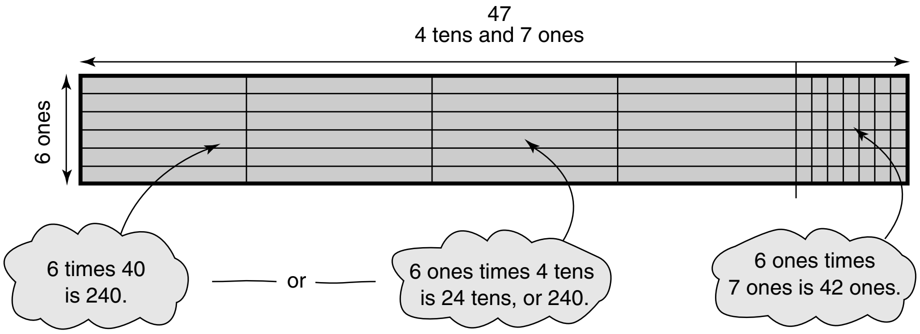


FIGURE 4.13

A rectangle filled with base-ten pieces is a useful model for two-digit-by-one-digit multiplication.

As shown in Figure 4.13, the rectangle can be “sliced” or separated into two parts so that one part will be 6 ones by 7 ones, or 42 ones, and the other will be 6 ones by 4 tens, or 24 tens. Notice that the base-ten language “6 ones times 4 tens is 24 tens” tells how many *pieces* (sticks of ten) are in the big section. To say “6 times 40 is 240” is also correct and tells how many units or square centimeters are in the section. Each section is referred to as a *partial product.* By adding the two partial products, you get the total product or area of the rectangle.

To avoid the tedium of drawing large rectangles and arranging base-ten pieces, use the base-ten grid paper found in the Blackline Masters. On the grid paper, students can easily draw accurate rectangles showing all of the pieces. Check to be sure students understand that for a product such as 74 × 8, there are two partial products, 70 × 8 = 560 and 4 × 8 = 32, and the sum of these is the product. Do not force any recording technique on students until they understand how to use the two dimensions of a rectangle to get a product.

Develop the Written Record

When the two partial products are written separately as in Figure 4.14(a), there is little new to learn. Students simply record the products and add them together. As illustrated, it is possible to teach students how to write the first product with a carried digit so that the combined product is written on one line. This traditional recording scheme is known to be problematic. The little carried digit is often the source of difficulty—it gets added in before the second multiplication or is forgotten.

There is absolutely no practical reason why students can’t be allowed to record both partial products and avoid the errors related to the carried digit. When you accept that, it makes no difference in which order the products are written. Why not simply permit students to do written multiplication as shown in Figure 4.14 without carrying? Furthermore, that is precisely how this is done mentally.

Most standard curricula progress from two digits to three digits with a single-digit multiplier. Students can make this progression easily. They still should be permitted to write all three partial products separately and not have to bother with carrying.

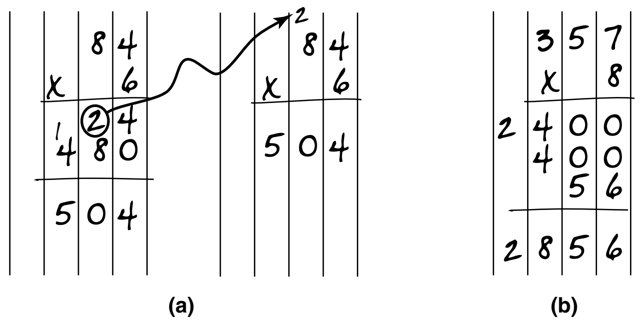


FIGURE 4.14

(a) In the standard form, the product of ones is recorded first. The tens digit of this first product can be written as a “carried” digit above the tens column. (b) It is quite reasonable to abandon the carried digit and permit the partial products to be recorded in any order.

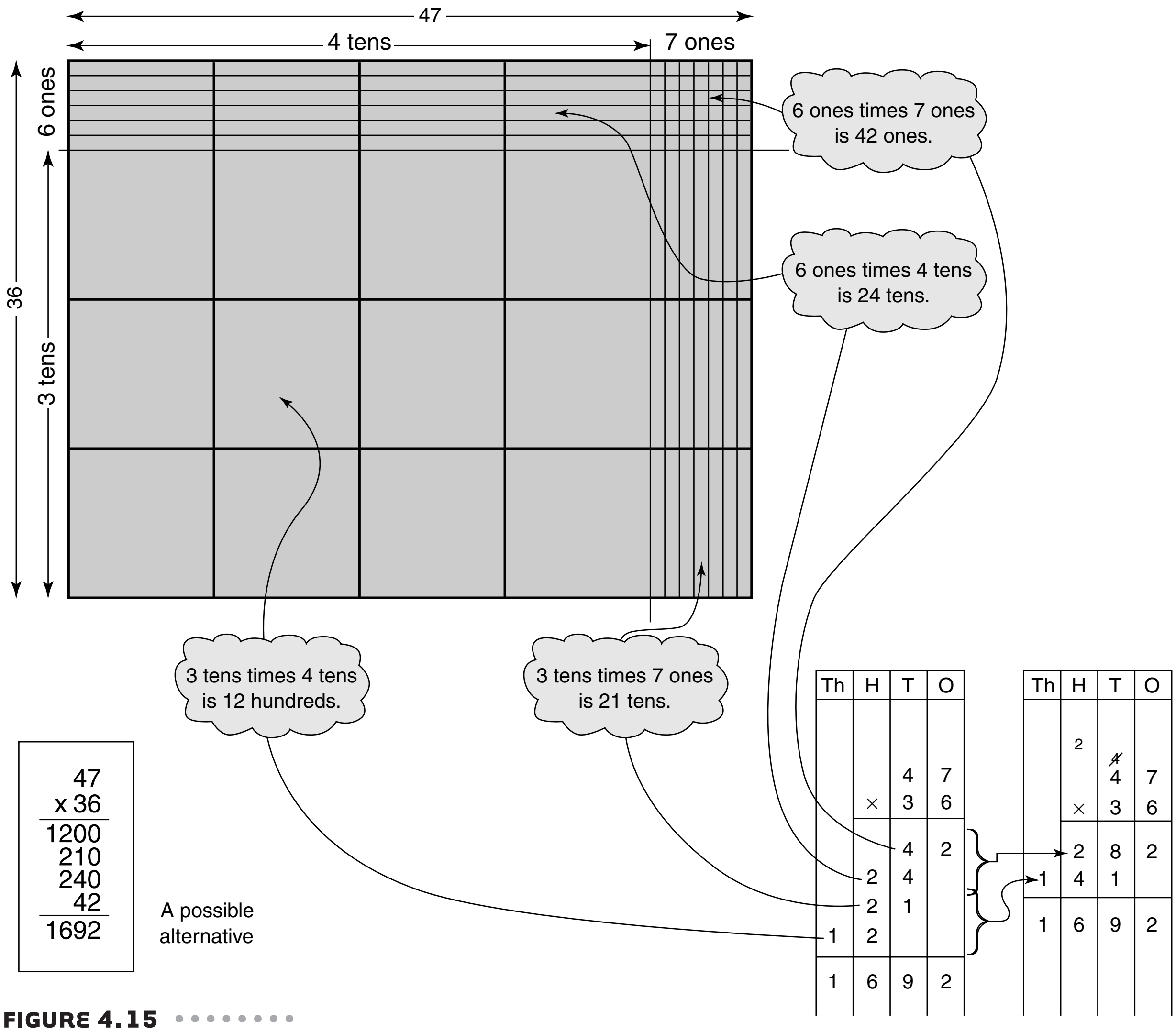


FIGURE 4.15

47 × 36 rectangle filled with base-ten pieces. Base-ten language connects the four partial products to the traditional written format. Note the possibility of recording the products in some other order.

Two-Digit Multipliers

With the area model, the progression to a two-digit multiplier is relatively straightforward. Rectangles can be drawn on base-ten grid paper, or full-sized rectangles can be filled in with base-ten pieces. There will be four partial products, corresponding to four different sections of the rectangle.

Figure 4.15 also shows the recording of four partial products in the traditional order and how these can be collapsed to two lines if carried digits are used. Here the second "carry" technically belongs in the hundreds column but it rarely is written there. Often it gets confused with the first and is thus an additional source of errors. The lower left of the figure shows the same computation with all four products written in a different order. This is quite an acceptable algorithm. In the rare instance when someone multiplies numbers such as 538 × 29 with pencil and paper, there would be six partial products. But far fewer errors would occur, requiring less instructional time and much less remediation.

Invented Strategies for Division

In our discussion of division facts (Chapter 3), we included something we called "near facts." In a near fact, the divisor and quotient are both less than ten but there is a remainder, as in 44 ÷ 8. Third- and fourth-grade students should have ample experiences with near facts. When these problems are expanded to those in which the quotients are more than 9 (e.g., 73 ÷ 6), the process evolves into invented strategies for division.

Sharing and Measurement Problems

Recall that there are two concepts of division. First there is the partition or fair-sharing idea, illustrated by this story problem:

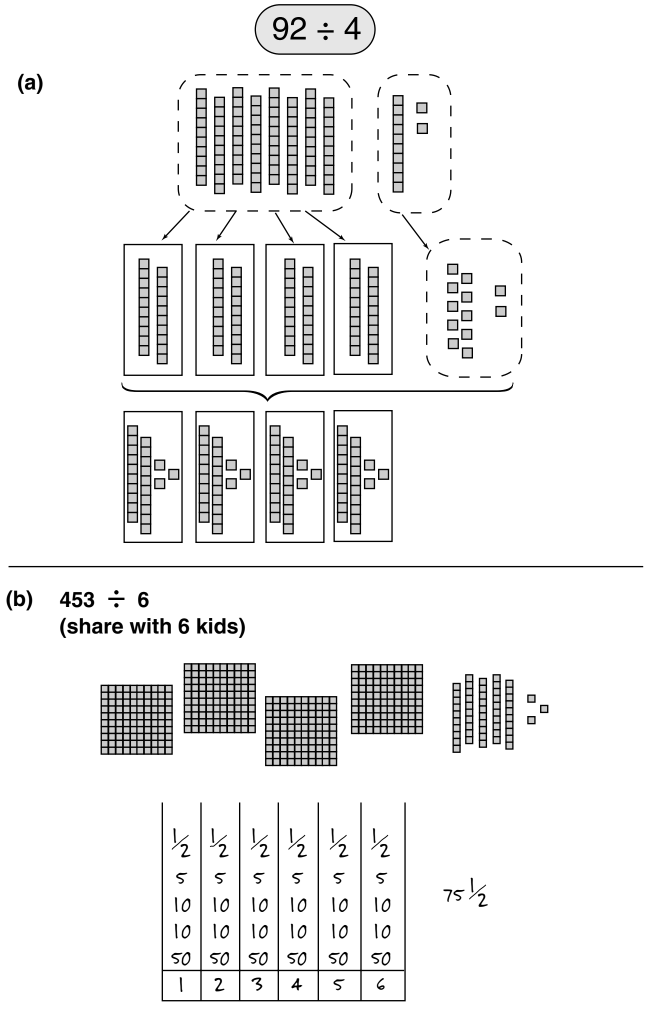
The bag has 783 jelly beans, and Aidan and her four friends want to share them equally. How many jelly beans will Aidan and each of her friends get?

Then there is the measurement or repeated subtraction concept:

Jumbo the elephant loves peanuts. His trainer has 625 peanuts. If he gives Jumbo 20 peanuts each day, how many days will the peanuts last?

Students should be challenged to solve both types of problems. However, the fair-share problems are often easier to solve with base-ten pieces. Furthermore, the traditional algorithm is built on this idea. Eventually, students will develop strategies that they will apply to both types of problem, even when the process does not match the action of the story.

Figure 4.16 shows some strategies that fourth-grade students have used to solve division problems. The first example illustrates 92 ÷ 4 using base-ten pieces and a sharing process. A ten is traded when no more tens can be passed out. Then the 12 ones are distributed, resulting in 23 in each set. This direct modeling approach with base-ten pieces is quite easy even for third-grade students to understand and use.



(c) **143 jelly beans shared with 8 kids**

Try 14 × 8 → 112
12 groups of 8 is 96.
12 groups in 100 leaves 4.
5 groups of 8 is 40.
And 3 more left over.
12 + 5 is 17 with 7 left.

FIGURE 4.16

Students use both models and symbols to solve division tasks.

Source: From *Developing Mathematical Ideas: Numbers and Operations, Part I: Building a System of Tens Casebook,* by D. Schifter, V. Bastable, & S. J. Russell. Copyright © 1999 by Education Development Center, Inc. Published by Dale Seymour Publications, an imprint of Pearson Learning. Used by permission.

In the second example, the student sets out the base-ten pieces and draws a "bar graph" with six columns. After noting that there are not enough hundreds for each kid, he mentally splits the 3 hundreds in half, putting 50 in each column. That leaves him with 1 hundred, 5 tens, and 3 ones. After trading the hundred for tens (now 15 tens), he gives 20 to each, recording 2 tens in each bar. Now he is left with 3 tens and 3 ones, or 33. He knows that 5 × 6 is 30, so he gives each kid 5, leaving him with 3. These he splits in half and writes $\frac{1}{2}$ in each column.

The student in the third example is solving a sharing problem but tries to do it as a measurement process. She wants to find out how many 8s are in 143. Initially she guesses. By multiplying 8 first by 10, then by 20, and then by 14, she knows the answer is more than 14 and less than 20. After some more work (not shown), she rethinks the problem as how many 8s in 100 and how many in 40.

Missing Factor Strategies

You can see in Figure 4.16 how the use of base-ten pieces tends to lead to a digit-by-digit strategy—share the hundreds first, then the tens, then the ones. Although this is precisely the conceptual background behind the traditional algorithm, it is digit oriented as opposed to an approach that helps students think of the whole value of the dividend. In Figure 4.16(c), the student is using a multiplicative approach. She is trying to find out, "What number times 8 will be close to 143 with less than 8 left over?" This is a good method to suggest to students in grades 3–5. It will build on their multiplication skills, it is a method that lends itself to mental estimation, and it can work quite well for most purposes.

STOP **Before reading further, consider the task of determining the quotient of 318 ÷ 7 by trying to figure out *what number times 7* (or *7 times what number*) is close to 318 without going over. Do not use the standard algorithm.**

There are several places to begin solving this problem. For instance, since 10 × 7 is 70 and 100 × 7 is 700, it has to be between 10 and 100, probably closer to 10. You might start adding up 70s:

70
\+ 70 is 140
\+ 70 is 210
\+ 70 is 280
\+ 70 is 350

So four 70s is not enough and five is too much. It has to be forty-something. At this point you could guess at numbers between 40 and 50. Or you might add on 7s. Or you could notice that forty 7s (280) leaves you with 20 plus 18 or 38. Or five 7s will be 35 of the 38 with 3 left over. In all, that's 40 + 5 or 45 with a remainder of 3.

Another starting point might be 50 × 7. This beginning likely indicates that 40 × 7 will be the largest multiple of ten.

This missing-factor approach is likely to be invented by some students if they are solving measurement problems such as the following:

Grace can put 6 pictures on one page of her photo album. If she has 82 pictures, how many pages will she need?

Alternatively, you can simply pose a task such as 82 ÷ 6 and ask students, "What number times 6 would be close to 82?" and continue from there.

Another approach to developing missing-factor strategies is to use cluster problems as discussed for multiplication. (See p. 117.) Here are two examples:

100 × 4	10 × 72
500 ÷ 4	5 × 70
4 × 25	2 × 72
6 × 4	4 × 72
527 ÷ 4	5 × 72
	381 ÷ 72

Notice that the missing-factor strategy is equally as good for one-digit divisors as for two-digit divisors. Also notice that it is okay to include division problems in the cluster. In the preceding example, 125 × 4 could easily have replaced 500 ÷ 4, and 400 ÷ 4 could replace 100 × 4. The idea is to keep multiplication and division as closely connected as possible.

Cluster problems accentuate a flexible approach to computation, helping students realize that there are many different good ways to compute. Another way to develop flexibility is to pose a division problem (or a multiplication problem) and have students solve the problem using two different approaches. Of course, neither of the methods should be the traditional algorithm or a calculator.

STOP **Solve 514 ÷ 8 in two different, nontraditional ways. Your ways may converge in similar places but begin with different first steps, or they may be completely different.**

Here are four possible starting points and there are certainly others:

10 × 8   400 ÷ 8   60 × 8   80 ÷ 8

Try to solve 514 ÷ 8 beginning with each of these starting points.

When students are first asked to solve problems using two methods, they often use a primitive or completely inefficient method for their second approach. For example, to solve 514 ÷ 8, a student might perform a very long string of subtractions (514 – 8 = 506, 506 – 8 = 498, 498 – 8 = 490, and so on) and count how many times he or she subtracted 8. Others will actually draw 514 tally marks and loop groups of 8. These students have not developed sufficient flexibility to think of other efficient methods. To help with this, pose problems along with two or three starting points and have students use each of the starting points to solve the problem. Your class discussions will help students begin to see more flexible approaches.

The Traditional Algorithm for Division

If you have been working along with the examples and approaches in this section, we hope you are convinced that students can use invented strategies for both one-digit divisors and two-digit divisors as long as the dividends are less than 1000 and a whole-number quotient with a remainder is all that is required. That is, it is not significantly faster to do 738 ÷ 43 by the traditional algorithm than to use a missing-factor approach. (Try it!) Notice that while doing the traditional algorithm you also have to do 308 ÷ 43, another problem as hard as the original. That is, the task often does not get easier as you go along. Compound this with the abundant difficulties of the traditional algorithm and the concomitant reteaching that inevitably takes place.

However, many will argue that students simply must have a more efficient method of dividing than those suggested here. Furthermore, if the curriculum requires division with decimal divisors or quotients to be carried out to get decimal results (in contrast to whole-number remainders), an argument can possibly be made for teaching a traditional algorithm. We, therefore, share with you one approach to the traditional long-division algorithm. Because the algorithm most often taught in textbooks is based on the partition or fair-sharing concept of division, that is the method described here. (Some teachers may want to explore a repeated subtraction algorithm that is very much like a missing-factor approach with partial products recorded in a column to the right of the division computation. See Figure 4.17 for an example.)

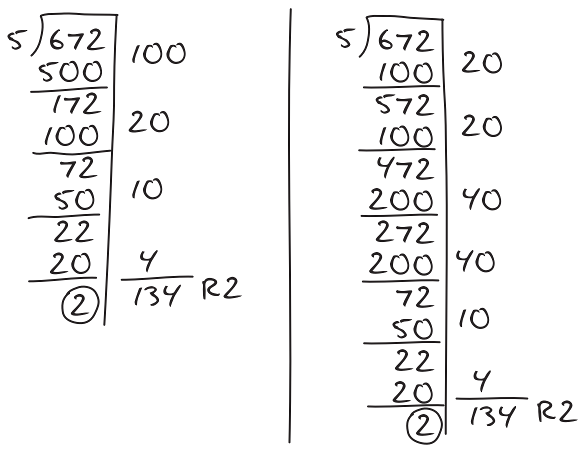


FIGURE 4.17

In the division algorithm shown, the numbers on the side indicate the quantity of the divisor being subtracted from the dividend. As the two examples indicate, the divisor can be subtracted from the dividend in any amount desired.

One-Digit Divisors

Typically, the division algorithm with one-digit divisors is introduced in the third grade. If done well, it should not have to be retaught, and it should provide the basis for two-digit divisors.

Begin with Models

Traditionally, for a problem such as 4)583, we might say "4 goes into 5 one time." This is quite mysterious to students. How can you just ignore the "83" and keep changing the problem? Preferably, you want students to think of the 583 as 5 hundreds, 8 tens, and 3 ones, not as the independent digits 5, 8, and 3. One idea is to use a context such as candy bundled in boxes of ten with 10 boxes to a carton. Then the problem becomes *We have 5 boxes, 8 cartons, and 3 pieces of candy to share between 4 schools evenly.* In this context, it is reasonable to share the cartons first until no more can be shared. Those remaining are "unpacked," and the boxes shared, and so on. Money ($100, $10, and $1) can be used in a similar manner.

Try the distributing or sharing process yourself using base-ten pieces (or draw squares, sticks, and dots). Use the problem 524 ÷ 3. Try to talk through the process without using "goes into." Think sharing.

Language plays an enormous role in thinking about the algorithm conceptually. Most adults are so accustomed to the "goes into" language that it is hard to let it go. For the problem 583 ÷ 4, here is some suggested language as you work through the task:

> *I want to share 5 hundreds, 8 tens, and 3 ones among these four sets. There are enough hundreds for each set to get 1 hundred. That leaves 1 hundred that I can't share.*
>
> *I'll trade the hundred for 10 tens. That gives me a total of 18 tens. I can give each set 4 tens and have 2 tens left over. Two tens is not enough to go around the four sets.*
>
> *I can trade the 2 tens for 20 ones and put those with the 3 ones I already had. That makes a total of 23 ones. I can give 5 ones in each of the four sets. That leaves me with 3 ones as a remainder. In all I gave out 1 hundred, 4 tens, and 5 ones with 3 left over.*

Develop the Written Record

The recording scheme for the long-division algorithm is not completely intuitive. You will need to be quite directive in helping children learn to record the fair sharing with models. There are essentially four steps:

1. *Share* and record the number of pieces put in each group.
2. *Record* the number of pieces shared in all. Multiply to find this number.
3. *Record* the number of pieces remaining. Subtract to find this number.
4. *Trade* (if necessary) for smaller pieces and combine with any that are there already. Record the new total number in the next column.

When students model problems with a one-digit divisor, steps 2 and 3 seem unnecessary. Explain that these steps really help when you don't have the pieces there to count.

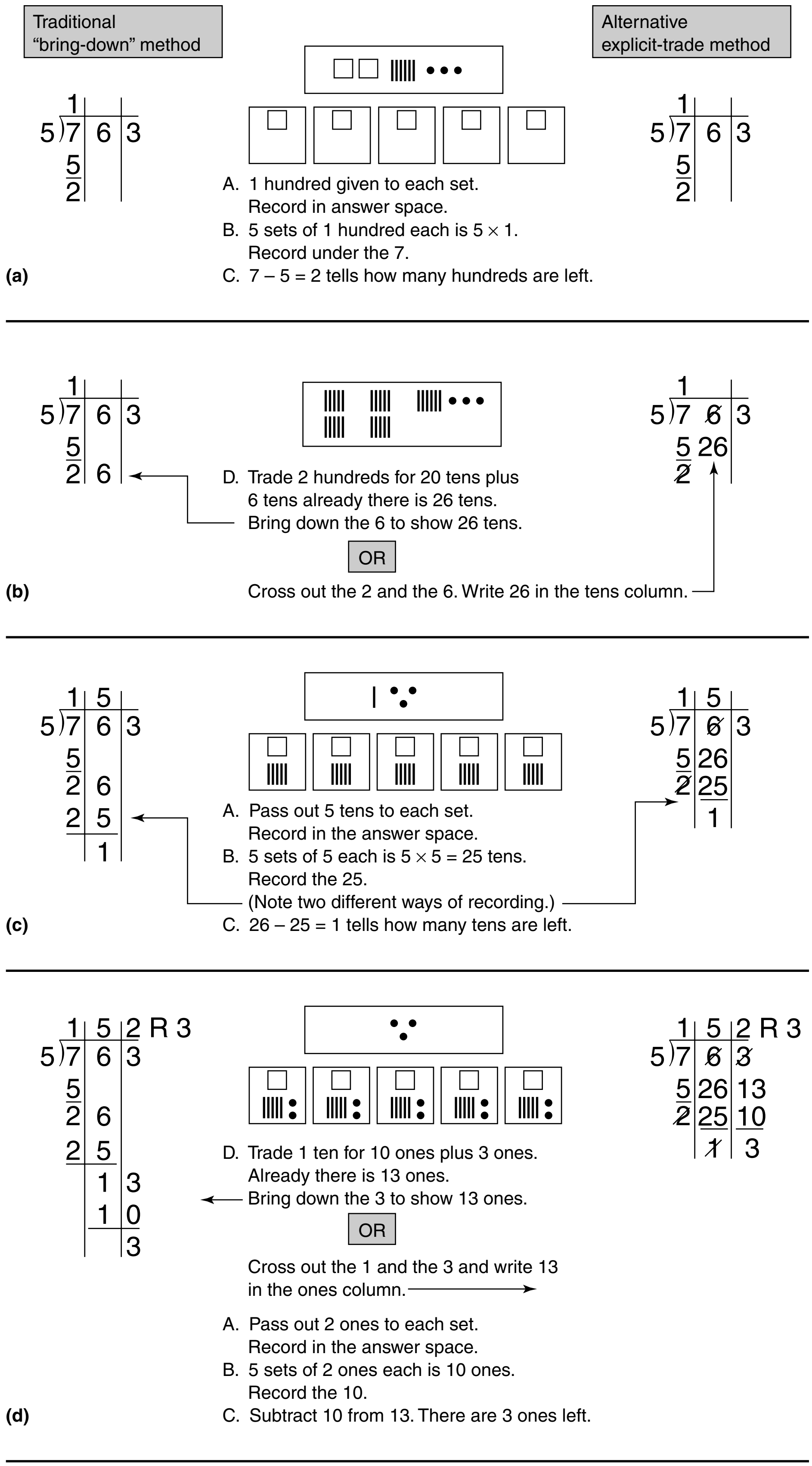
Record Explicit Trades

Figure 4.18 details each step of the recording process just described. On the left, you see the traditional algorithm. To the right is a suggestion that matches the actual action with the models by explicitly recording the trades. Instead of the somewhat mysterious "bring-down" procedure, the traded pieces are crossed out, as is the number of existing pieces in the next column. The combined number of pieces is written in this column using a two-digit number. In the example, 2 hundreds are traded for 20 tens, combined with the 6 that were there for a total of 26 tens. The 26 is therefore written in the tens column.

Students who are required to make sense of the long-division procedure find the explicit-trade method easier to follow. It is important to spread out the digits in the dividend when writing down the problem. (The explicit-trade method is a Van de Walle invention. It has been used successfully in grades 3 to 8. You will not find it in textbooks.)

FIGURE 4.18

The traditional and explicit-trade methods are connected to each step of the division process. Every step can and should make sense.



Both the explicit-trade method and the use of place-value columns will help with the problem of leaving out a middle zero in a problem (see Figure 4.19).

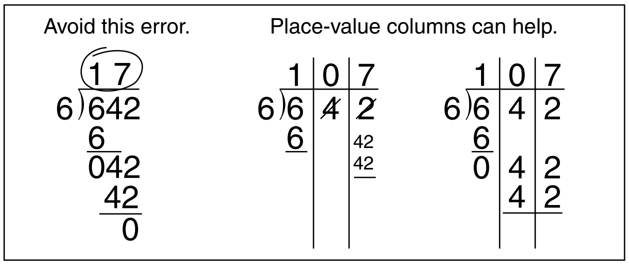


FIGURE 4.19

Using lines to mark place-value columns can help avoid forgetting to record zeros.

Two-Digit Divisors

There is almost no justification for having students master the division algorithm with two-digit divisors. A large chunk of the fourth, fifth, and sometimes sixth grade is frequently spent on this outdated skill. The cost in terms of time and students' attitudes toward mathematics is enormous. Only a few times in any adult's life will an exact result to such a computation be required and a calculator not be available. If you can possibly influence the removal of this outdated skill from your school's curriculum, you are encouraged to speak up.

With a two-digit divisor, it is hard to come up with the right amount to share at each step. A guess too high or too low means you have to erase and start all over.

An Intuitive Idea

Suppose that you were sharing a large pile of candy with 36 friends. Instead of passing them out one at a time, you conservatively estimate that each person could get at least 6 pieces. So you give 6 to each of your friends. Now you find there are more than 36 pieces left. Do you have everyone give back the 6 pieces so you can then give them 7 or 8? That would be silly! You simply pass out more.

The candy example gives us two good ideas for sharing in long division. First, always underestimate how much can be shared. You can always pass out some more. Second, if there is enough left to share some more, just do it! To avoid ever overestimating, always pretend there are more sets among which to share than there really are. For example, if you are dividing 312 by 43 (sharing among 43 sets or "friends"), pretend you have 50 sets instead. Round *up* to the next multiple of 10. You can easily determine that 6 pieces can be shared among 50 sets because 6 × 50 is an easy product. Therefore, since there are really only 43 sets, clearly you can give *at least* 6 to each. Always consider a larger divisor; *always round up*. If your underestimate leaves you with more to share, simply pass out some more.

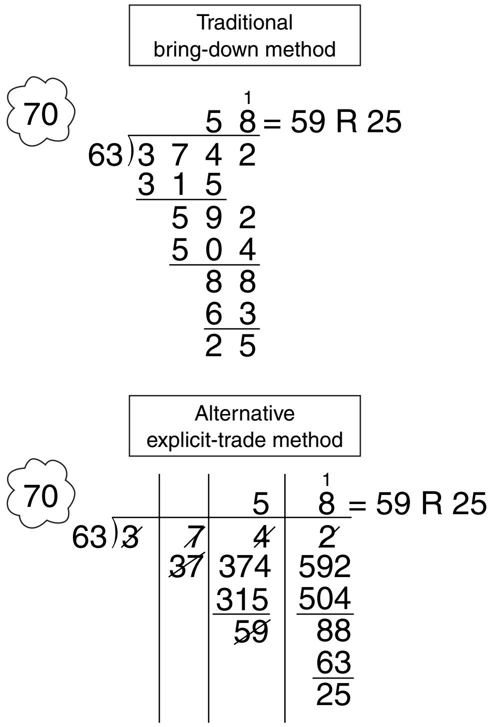


FIGURE 4.20

Round the divisor up to 70 to think with, but multiply what you share by 63. In the ones column, share 8 with each set. Oops! 88 left over, just give 1 more to each set.

Using the Idea Symbolically

These ideas are used in Figure 4.20. Both the traditional method and the explicit-trade method of recording are illustrated. The rounded-up divisor, 70, is written in a little "think bubble" above the real divisor. Rounding up has another advantage: It is easy to run through the multiples of 70 and compare them to 374.

Work through the problem one step at a time, saying exactly what each recorded step stands for.

Always rounding the divisor up has two advantages. It reduces the mental strain of making choices and essentially eliminates the need to erase. If an estimate is too low, that's okay. And if you always round up, the estimate will never be too high. Nor is there any reason ever to change to the more familiar approach. It is just as good for adults as for children. The same is true of the explicit-trade notation. It is certainly an idea to consider.

Assessment Note

Parents are perhaps more interested in their children's computational skills than in any other area. When students do well on computation tests, parents are pleased. But what do you know when students do not do well? At best you can make inferences based on the papers turned in. You can look for basic-fact errors and carelessness or perhaps find a systematic error in an algorithm. What you do not know is how students are solving these problems and what ideas and strategies they have developed that are useful or need further development.

When computational strategies and algorithms are developed in the manner suggested in this chapter, every day you are presented with a wealth of assessment data. The important thing is to gather, record, and use these data for individual children the same as you would for tests and quizzes. A simple chart something like the one in Figure 4.21 may be all you need. Note that the third column includes a minirubric or a three-point scale. Students' names can be arranged in groups, by how they sit in the room, or alphabetically—any way that makes them easy to find.

As you walk around in the during portion of your lessons, and also in the after portion when students explain their computation strategies and reasoning, you can make notes on the chart. Make a new chart each week, but keep the old ones to provide evidence of growth over time. These charts can be useful for grading and for parent conferences. There is no harm in giving an occasional quiz or test of computational skills. But avoid giving more value to tests simply because they are objective.

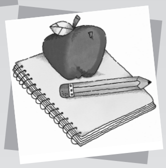
FIGURE 4.21

A checklist with space for comments or notes lets you record daily observations of students' direct modeling and invented strategies.

Topic: Mental addition and subtraction

Student	Adds 2-digit + 1-digit numbers	Adds 2-digit + 2-digit numbers; note methods	Flexibility in choosing a method: 1, 2, 3	Comments
Lalie				
Pete				
Sid				
Lakeshia				

EXPANDED LESSON



Area Model for Multiplication

GRADE LEVEL: Fourth or fifth grade.

MATHEMATICS GOALS

To develop strategies for two-digit multiplication using the area model.

THINKING ABOUT THE STUDENTS

Students have mastered most of their basic multiplication facts and understand that multiplication can be thought of as repeated addition. They have multiplied numbers by multiples of 10. They understand that length times width gives the area of a rectangle.

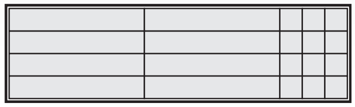
MATERIALS AND PREPARATION

- Accurately cut from poster board a rectangle that is 47 cm by 36 cm. Use this to trace rectangles on large sheets of paper, one for each group of two or three students.
- Provide each group of students with base-ten materials, enough to fill the rectangle.
- Draw a 23 cm x 4 cm rectangle on a transparency.
- Overhead base-ten materials or regular base-ten materials.
- *Note:* If your base-ten models are not based on centimeters, adjust all rectangles to match the size of your materials.

lesson

BEFORE

- Show the transparency of the 23 cm × 4 cm rectangle and write the dimensions of all four sides. Explain that you want to find out how many unit squares (show some) will fit in the rectangle, but you don't have that many unit pieces. *How else could we fill the rectangle?* When students suggest the use of tens pieces, have them work quickly in pairs to decide how many tens pieces and how many ones will fit in the rectangle.
- Position 8 tens and 12 ones inside the rectangle. Ask: *Now can we tell how many small squares are in the rectangle?* Again, have students solve this quickly and share their solutions (80 and 12, or 92 units in all).



The Task

- Determine the area of a 47 cm × 36 cm rectangle.

Establish Expectations

- Recognize counting the squares one by one as a legitimate way to determine the area of the rectangle, but explain to students that they need to find a quicker way to determine the number of squares inside the rectangle.
- Solutions must include a drawing, numbers, and explanation of how students determined the total.

DURING

- Most students will first fill the rectangle with as many hundreds pieces as possible. If some students use only the small ones pieces, suggest that they might try using large pieces.
- Once students have filled their rectangle, they should work to add up the total number of ones. Observe the way students count the pieces. Expect that students will count the individual pieces rather than use the dimensions of the rectangle in a multiplication.
- For students who have solved the problem and completed their written explanation, see if they can connect what they have done to the numbers in the rectangle dimensions.

AFTER

- Begin by recording (without comment) all answers to the task. It is quite possible that not all groups will have gotten 1692.
- Have students share strategies. Begin with students who may have used less than efficient methods. Try to include students who wrote down and added the four partial products (1200, 210, 240, and 42). These four partial products most directly relate to the standard algorithm and are useful for invented strategies as well.
- As new strategies emerge, ask students to compare and contrast the new strategies to ones already shared. Do not evaluate any approach or answer.

ASSESSMENT NOTES

- Do students see the efficiency of using the larger hundreds pieces? Do they see and use four separate sections to the filled-in rectangle?
- Do students make connections between the dimensions of the rectangle and its area? Do they seem aware of the connection between multiplication and area?
- Can students use multiples of ten to determine the smaller regions?

next steps

- This task is profitably repeated using rectangles drawn on base-ten grid paper (see Blackline Master 16).
- For students who have difficulty with this task, use cm-grid paper (see Blackline Master 8) to draw a 15 x 30 rectangle. Have students use base-ten materials to fill inside the rectangle to determine the area. Hundreds pieces fit into this region so that students have to deal with only one narrow region left uncovered.
- When students are clearly using four partial products in their solutions, challenge them to connect their strategies to the dimensions of the rectangle and then see if they can determine the area of a 64 x 73 rectangle without using a drawing.